The Contact Dynamics method: A nonsmooth story

Frédéric Dubois, Vincent Acary, Michel Jean

C. R. Mecanique ••• (••••) •••–•••

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The legacy of Jean-Jacques Moreau in mechanics

The Contact Dynamics method: A nonsmooth story

La méthode de la dynamique des contacts, histoire d'une mécanique non régulière

Frédéric Dubois [a,b,*], Vincent Acary [d], Michel Jean [c]

[a] *LMGC, Univ. Montpellier, CNRS, Montpellier, France*
[b] *MIST, Univ. Montpellier, CNRS, IRSN, Montpellier, France*
[c] *Aix-Marseille Université, CNRS, Centrale Marseille, Marseille, France*
[d] *LJK, INRIA, Université de Grenoble Alpes, Grenoble, France*

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ABSTRACT

When velocity jumps are occurring, the dynamics is said to be nonsmooth. For instance, in collections of contacting rigid bodies, jumps are caused by shocks and dry friction. Without compliance at the interface, contact laws are not only non-differentiable in the usual sense but also multi-valued. Modeling contacting bodies is of interest in order to understand the behavior of numerous mechanical systems such as flexible multi-body systems, granular materials or masonry. These granular materials behave puzzlingly either like a solid or a fluid and a description in the frame of classical continuous mechanics would be welcome though far to be satisfactory nowadays. Jean-Jacques Moreau greatly contributed to convex analysis, functions of bounded variations, differential measure theory, sweeping process theory, definitive mathematical tools to deal with *nonsmooth* dynamics. He converted all these underlying theoretical ideas into an original nonsmooth implicit numerical method called Contact Dynamics (CD); a robust and efficient method to simulate large collections of bodies with frictional contacts and impacts. The CD method offers a very interesting complementary alternative to the family of smoothed explicit numerical methods, often called Distinct Elements Method (DEM). In this paper developments and improvements of the CD method are presented together with a critical comparative review of advantages and drawbacks of both approaches.

RÉSUMÉ

Lorsque des sauts de vitesse se produisent, la dynamique est dite non régulière. Par exemple, dans les collections de solides supposés rigides rentrant en contact, les sauts sont causés par les chocs et le frottement sec. L'absence de déformabilité fait que les lois de contact sont, non seulement non différentiables au sens usuel, mais aussi multi-valuées. Élaborer des modèles de solides en contact est un moyen de comprendre le comportement de nombreux systèmes mécaniques tels que les systèmes multi-corps flexibles, les matériaux granulaires ou les maçonneries. Les matériaux granulaires se comportent de manière étrange, soit comme des solides, soit comme des fluides, et une

* Corresponding author.
E-mail addresses: frederic.dubois@umontpellier.fr (F. Dubois), vincent.acary@inria.fr (V. Acary), mjean.recherche@wanadoo.fr (M. Jean).

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description dans le cadre de la mécanique classique des milieux continus, qui serait souhaitable, est loin d'être encore satisfaisante. Jean-Jacques Moreau a contribué, de façon fondamentale, à l'analyse convexe, à la théorie des fonctions à variations bornées et des mesures différentielles ainsi qu'au processus de rafle, outils mathématiques décisifs pour traiter *la dynamique non régulière*. Il a converti ces idées théoriques sous-jacentes en une méthode numérique originale appelée *Contact Dynamics* (CD), qui est une méthode non régulière implicite et aussi une méthode robuste et efficace pour simuler de larges collections de solides avec du contact frottant et des impacts. Le méthode CD offre une alternative très intéressante à la famille de méthodes usuelles régularisées explicites, comme la méthode des éléments distincts (DEM). Dans cet article, des développements et des perfectionnements de la méthode CD sont présentés ainsi qu'une étude critique comparative des avantages et inconvénients des deux approches.

1. Introduction

The generic label (DEM) Discrete Elements Methods, also called Distinct Elements Methods, refers to methods that, oppositely to Finite Elements Methods (FEM) dedicated to the description of media in the frame of continuous mechanics, consider a sample as an assembly of distinct bodies. Nowadays, such methods are widely used in the numerical modeling of divided materials and structures. Natural applications concern the simulation of granular materials, suspensions, fractured materials, masonries, rock mass, etc. in domains such as geophysics, mining, chemical engineering, civil engineering, biomechanics, etc. DEM are also used for their capability to represent the various states of a collection of solids (gas, fluid, solid) and to represent some phase changes (solid to solids and vice versa) in the spirit of meshless methods [1] or particle methods [2,3]. But such methods are also applied in multi-body systems such as mechanisms and robotics. Usually, in such methods, one considers collections of rigid bodies, subject to interaction laws such as frictional contact laws that are steep laws.

A number of leading methods are derived from the pioneering work of Cundall [4] actually referred to the generic label DEM. This work may be considered also as a modification of the genuine Molecular Dynamics method as proposed by Allen and Tidsley [5]. Since such methods are particularly pragmatic, steep frictional contact laws are modeled as nonlinear laws using some regularization techniques, and explicit time integrators are used to cope with the nonlinear behavior. At the end, such methodology leads to a set of uncoupled linear equations that can be solved straightforwardly. The name DEM is commonly referring to those *smoothed and explicit* methods. The weaknesses of such methods come from their principle: small time steps are mandatory due to explicit time integrators, the choice of relevant parameters may be tricky, since they are used to manage several phenomena: contact condition, macroscopic mechanical response, dynamical properties, etc. In particular, in order to ensure the stability of explicit schemes, it is necessary to introduce some damping, either generated by the frictional contact laws, either as a numerical trick. There exist also several "confidential" methods such as Discontinuous Deformation Analysis (DDA) [6,7], or Discrete Fracture Network (DFN) [8]. For a small number of objects, the classical Finite Element Method (FEM) can also manage contact problems [9].

Jean-Jacques Moreau introduced the (CD) Contact Dynamics method during the year 1984. It is inspired by a formulation of unilateral contact, shock laws, Coulomb friction, through Convex Analysis. The contacting laws are thus non differentiable steep laws. They are managed with an implicit method using a Non-Linear Gauss–Seidel algorithm (NLGS) at each step. These laws account roughly for the main features of contact and friction and are relevant in multi-bodies collections where sophisticated laws cannot be exhibited for sure. The method uses large time steps, but each time step is time consuming. So oppositely to the above smoothed and explicit DEM method, the CD method is a *nonsmooth and implicit* method. Note that implicit methods enable to compute correctly equilibrium states, which is not always the case with explicit methods. The method can also conserve, with a suitable choice of parameters, the total energy of the system in discrete time. In this paper, we shall discuss the pros and cons of the CD method with respect to classical smoothed DEM.

2. Moreau contribution to Contact Dynamics genesis

First, it is to be remembered that Jean-Jacques Moreau was earlier concerned with fluid mechanics. He left an original result concerning the helicity invariant in perfect fluid dynamics (1961) [10].

Jean-Jacques Moreau was introducing himself as involved in mechanics, claiming that he was using mathematics just enough for his mechanical purpose. Actually he was the author of highly sophisticated concepts in mathematics mainly in the field of measure theory. He developed definitely the theory of locally bounded variation functions [11], the proper mathematical setting for the nonsmooth dynamics and the sweeping process theory [12–15]. After the works of the pioneers, H. Minkowski and M.W. Fenchel, he set up Convex Analysis, a theory also developed at the same time by R.T. Rockafellar; see for instance the numerous references in the famous book [16] to the results of Jean-Jacques Moreau. Convex Analysis is the proper tool to deal with nonsmooth mechanics i.e. mechanics where the behavior laws are not differentiable in the

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usual sense [17–19]. Elastoplasticity mechanics is an example. Another example is the mechanics of frictional collisions between rigid bodies, a problem where velocity jumps are expected. The first-order sweeping process introduced by Jean-Jacques Moreau, motivated by the quasi-static evolution of elastoplastic systems [20,21], seems to have provided one of the first occurrence of measure differential inclusions in the literature, with the work of M. Schatzman [22]. For the application to nonsmooth dynamics, the second-order sweeping process is a fundamental tool for the mathematical analysis and the design of numerical schemes. The formulation of the Signorini condition at the velocity level is crucial in the development of the CD method. This condition may be viewed as the time derivative of the unilateral constraints. Firstly, it allows one to get some very good dissipation properties ensuring stability. This lies in the fact that it reduces the index of the associated switched differential algebraic equation. In other words, the unilateral constraints is written as a relation between the velocity and the impulse. In this way, the mechanical power of the system is described in a consistent way. Secondly, the second-order sweeping process is a direct extension of the Newton impact law, written for the first time in a numerical tractable way. Some authors call it the "Moreau impact law", since it is really an outstanding contribution of Jean-Jacques Moreau. For the numerical practice, it also ensures the Newton impact law to be satisfied in discrete time. Finally, another major contribution of Jean-Jacques Moreau is the formulation of frictional impact laws in terms of Convex Analysis, with equivalent formulations, for instance in terms of differential inclusions, pseudo-potentials of dissipation, and variational inequalities.

At the same time, other peoples contributed to the development of the method and its applications. An extension of the CD method to deformable bodies has been proposed by M. Jean under the name of Nonsmooth Contact Dynamics (NSCD). He proposed also a template of numerical architecture inspired by the idea of a mechanical scheme as two pairs of spaces in duality (global variables space, local variables space), see the next section 3. The LMGC90[1] open-source software has been developed around this skeleton by Frédéric Dubois et al. and enriched by all kinds of contacting rigid or deformable objects with a variety of interaction laws.

Currently the method is still developed by its precursors, but some relevant contributions are noticeable. The theory of bi-potential has been proposed by Gery de Saxcé. It is in the spirit of Convex Analysis and uses the same kind of frictional contact laws than the CD method. It comes out with a nice property, the reaction force appearing as the projection of a linear combination of the reaction with the relative velocity onto Coulomb's cone [23]. Another way of viewing this method is that it may come out as a variational inequality over a cone and not a quasi-variational inequality. Glocker [24] has proposed more sophisticated nonsmooth contact laws, also in the spirit of Convex Analysis, for evolutions where relative velocities and accelerations have jumps. The results may be quite accurate for multi-bodies systems with few degrees of freedoms. Brogliato and Acary [25–28] have proposed several improvements either concerning impact laws, enhanced time-stepping schemes or the extension of the non-smooth framework to new application fields as electrical circuits. There are also many methods and algorithms dedicated to quasi-static evolutions. We restrict the present paper to nonsmooth dynamics.

3. Current appraisal

We shortly give a hint of how the CD method works. Details may be found in the original papers by Jean-Jacques Moreau [29–31], M. Jean [32], and in the books [33,34].

3.1. A robust framework

As explained by Moreau contact problems in dynamics are non-smooth, due to 1/ kinematic unilateral constraints that introduce nonsmoothness in space, 2/ collisions that are expected to induce velocity jumps, which lead to nonsmoothness in time, 3/ multi-valued mappings between contact reactions and the contact velocities that introduce nonsmoothness in the law. Following the spirit of the sweeping process, the dynamical problem is written as a measure differential inclusion; it allows us to integrate and discretize non-smooth dynamical systems [35,36].

3.1.1. Dynamics of rigid bodies

In the following the rigid-body motion is governed by a Newton–Euler system of non-smooth equations:

$$\begin{cases} \mathbb{M}\,\mathrm{d}\mathbf{v} &= \mathbf{F}_{\mathrm{ext}}(t)\,\mathrm{d}t + \mathrm{d}\mathbf{I} \\ \mathbb{J}\,\mathrm{d}\boldsymbol{\Omega} &= -\boldsymbol{\Omega}\times\mathbb{J}\boldsymbol{\Omega} + \mathbf{M}_{\mathrm{ext}}(t)\,\mathrm{d}t + \mathrm{d}\mathbf{M} \end{cases} \tag{1}$$

where:

- $\mathrm{d}\mathbf{v}$ is the differential measure associated with velocity $\mathbf{v}$, considered as a time function of Bounded Variations (BV) and $\mathrm{d}\boldsymbol{\Omega}$ the differential measure of spin $\boldsymbol{\Omega}$ expressed in a frame attached to the body,
- $\mathrm{d}t$ is the Lebesgue measure,
- $\mathbb{M}$ and $\mathbb{J}$ represent respectively the mass and the inertia matrices,

[1] https://git-xen.lmgc.univ-montp2.fr/lmgc90/lmgc90_user.

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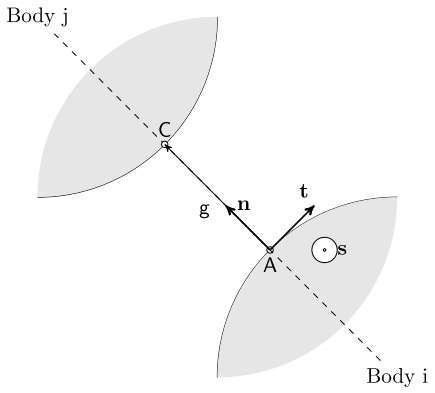

Fig. 1. Contact frame.

- $\mathbf{F}_{\text{ext}}(t)$ and $\mathbf{M}_{\text{ext}}(t)$ are the force and torque resultants due to external loads,
- $\mathrm{d}\mathbf{I}$ and $\mathrm{d}\mathbf{M}$ are the impulse and angular impulse measures due to contact.

Remark 1. From a practical point of view, one writes the Euler equation in a body-fixed frame which makes $\mathbb{J}$ diagonal. However, except in special cases (sphere, cube, etc.), when $\mathbb{J}\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}$ are collinear, the Euler equation remains non-linear and depends implicitly on the orientation of objects.

If f is a right continuous locally BV function, the measure of $]t_{\text{b}}, t_{\text{e}}]$ by $\mathrm{d}f$ is given by

$$\int_{]t_{\text{b}},t_{\text{e}}]} \mathrm{d}f = f(t_{\text{e}}) - f(t_{\text{b}}) \tag{2}$$

Recall that every function of bounded variations admits left and right limits: f^- and f^+. We choose to identify the BV function f as its right limit f^+. Therefore, velocity and spin can be obtained such that $\mathbf{v}(t_{\text{e}}) = \mathbf{v}(t_{\text{b}}) + \int_{]t_{\text{b}},t_{\text{e}}]} \mathrm{d}\mathbf{v}$ and $\boldsymbol{\Omega}(t_{\text{e}}) = \boldsymbol{\Omega}(t_{\text{b}}) + \int_{]t_{\text{b}},t_{\text{e}}]} \mathrm{d}\boldsymbol{\Omega}$. The position is computed using $\mathbf{x}(t_{\text{e}}) = \mathbf{x}(t_{\text{b}}) + \int_{t_{\text{b}}}^{t_{\text{e}}} \mathbf{v}(t)\,\mathrm{d}t$. The orientation is computed by integrating with respect to the Lebesgue measure $\dot{\mathcal{R}} = \mathcal{R}\tilde{\boldsymbol{\Omega}}$ where $\tilde{\boldsymbol{\Omega}}x = \boldsymbol{\Omega} \times x$.

In the following, $\mathbf{V} = \{\mathbf{v}, \boldsymbol{\Omega}\}$ is the generalized velocity, $\mathbf{q} = \{\mathbf{x}, \mathcal{R}\}$ is the generalized coordinates, $\bar{\mathbb{M}} = \mathrm{diag}(\mathbb{M}, \mathbb{J})$ is the generalized inertia matrix, $\bar{\mathbf{F}}_{\text{ext}}(t) = \{\mathbf{F}_{\text{ext}}(t), \mathbf{M}_{\text{ext}}(t)\}$, $\bar{\mathbf{F}}_{\text{quad}}(t) = \{0, -\boldsymbol{\Omega} \times \mathbb{J}\boldsymbol{\Omega}\}$ and $\bar{\mathbf{I}}_{]t_{\text{b}},t_{\text{e}}]} = \{\int_{]t_{\text{b}},t_{\text{e}}]} \mathrm{d}\mathbf{I}, \int_{]t_{\text{b}},t_{\text{e}}]} \mathrm{d}\mathbf{M}\}$ are respectively external, quadratic and contact contributions. Finally, the system takes the form:

$$\bar{\mathbb{M}}(\mathbf{V}(t_{\text{e}}) - \mathbf{V}(t_{\text{b}})) = \int_{t_{\text{b}}}^{t_{\text{e}}} \bar{\mathbf{F}}_{\text{ext}}(t)\,\mathrm{d}t + \int_{t_{\text{b}}}^{t_{\text{e}}} \bar{\mathbf{F}}_{\text{quad}}(t)\,\mathrm{d}t + \bar{\mathbf{I}}_{]t_{\text{b}},t_{\text{e}}]} \tag{3}$$

For simplicity sake the symbol $\bar{}$ is omitted in the following. Note that the time derivative of $\mathbf{q}$ is not directly related to the velocity $\mathbf{V}$ since $\dot{\mathcal{R}} = \mathcal{R}\tilde{\boldsymbol{\Omega}}$. In the sequel, we shall write $\dot{\mathbf{q}} = T(\mathbf{q})\mathbf{V}$. The expression (3) is no more than a balance of momentum over the time interval $]t_{\text{b}}, t_{\text{e}}]$. It is nonlinear due to contact and to $\mathbf{F}_{\text{quad}}(t)$.

3.1.2. Dynamics of deformable bodies

We mean by deformable bodies, bodies described in the frame of continuous mechanics. When numerical computation is concerned, those bodies are discretized through some finite elements techniques, so that the degrees of freedom are commonly the nodes coordinates. For instance, for a viscoelastic body within the small perturbations assumptions, the linear governing dynamical equation is:

$$\mathbb{M}\,\mathrm{d}\mathbf{v} + \mathbb{C}\mathbf{v}\,\mathrm{d}t + \mathbb{K}\mathbf{x}\,\mathrm{d}t = \mathbf{F}_{\text{ext}}(t)\,\mathrm{d}t + \mathrm{d}\mathbf{I} \tag{4}$$

where $\mathbf{x}$ is the coordinate with respect to a reference configuration, $\mathbb{C}$ is the viscosity matrix, and $\mathbb{K}$ is the stiffness matrix. In the case of large deformations or nonlinear behavior, one can use an updated Lagrange formulation nesting the linearized previous approach in Newton–Raphson loops.

3.1.3. Contact kinematics

Candidates to contact (shortly contacts) labeled α are selected. By candidates to contact is meant a pair of bodies close enough according to some criteria so that the reaction between those bodies (vanishing or not) has to be computed. For any candidates to contact α between bodies i and j, one defines: the gap $g^\alpha = D(\mathbf{q}_i, \mathbf{q}_j)$ (the signed distance from

body i to body j) and the local frame $\{\mathrm{A}, \mathbf{t}, \mathbf{n}, \mathbf{s}\}$; see Fig. 1. For the sake of simplicity, the contact locus is considered as punctual. It is the case when strictly convex boundary are meeting. Knowing the nearest points C and A, it follows that $D(\mathbf{q}_i, \mathbf{q}_j) = (\mathbf{x}_\mathrm{C} - \mathbf{x}_\mathrm{A}) \cdot \mathbf{n}$. In the case of meshed objects, one has to consider contact elements composed of a node versus a face of the antagonist discretized boundaries. There are more sophisticated contact elements.

For a given contact α, obvious kinematic relations allow one to express the relative velocity $\boldsymbol{\mathcal{V}}^\alpha$ at contact as a function of the generalized velocities $\mathbf{V}^{ij}$ of contacting bodies i and j:

$$\boldsymbol{\mathcal{V}}^\alpha = \mathbb{H}^{\alpha,\top}(\mathbf{q}_i, \mathbf{q}_j)\mathbf{V}^{ij} \tag{5}$$

where $\mathbf{V}^{ij} = [\mathbf{V}^{i,\top}, \mathbf{V}^{j,\top}]^\top$. Obviously, $D(\cdot)$ and $\mathbb{H}^{\alpha,\top}(\cdot)$ depend on the positions $\mathbf{q}_i$ and $\mathbf{q}_j$ of bodies i and j, which are not known *a priori* since they are solutions to the problem. Thus computing more accurately $\mathbb{H}^{\alpha,\top}(\mathbf{q}_i, \mathbf{q}_j)$ is a nonlinear problem.

Using duality considerations (equality of power expressed in terms of global or local variables),

$$\begin{array}{ccc} \mathbf{q}, \mathbf{V} & \leftarrow \text{ Equations of motion } \rightarrow & \mathbf{I} \\ \mathbb{H}^\top \downarrow & & \uparrow \mathbb{H} \\ \mathbf{g}, \boldsymbol{\mathcal{V}} & \leftarrow \text{ Interaction laws } \rightarrow & \boldsymbol{\mathcal{I}} \end{array} \tag{6}$$

the local contact impulse may be mapped into a generalized impulse:

$$\mathbf{I}^{ij} = \mathbb{H}^\alpha(\mathbf{q}_i, \mathbf{q}_j)\boldsymbol{\mathcal{I}}^\alpha \tag{7}$$

where $\mathbb{H}^{\alpha,\top}(\mathbf{q}_i, \mathbf{q}_j)$ is the transpose of $\mathbb{H}^\alpha(\mathbf{q}_i, \mathbf{q}_j)$ and $\mathbf{I}^{ij} = [\mathbf{I}^{i,\top}, \mathbf{I}^{j,\top}]^\top$. Another important relation relates the normal component of the relative velocity to the gap,

$$\dot{g}^\alpha = \mathcal{V}_\mathrm{N}^\alpha \tag{8}$$

which means that at the contact α the time derivative of the gap is the normal component of the relative velocity.

3.1.4. Frictional contact laws

We present here two basic nonsmooth laws that are used in the kernel of the CD method. Other more sophisticated frictional contact laws may be derived from these seed laws as mentioned later on (section 4.1). Consider for a moment that the reaction exerted between two contacting bodies at the contact α is a force $\boldsymbol{\mathcal{R}}^\alpha$ and $\boldsymbol{\mathcal{V}}^\alpha$ is the relative velocity. $\boldsymbol{\mathcal{R}}^\alpha$ has components on the local frame $\mathcal{R}_\mathrm{T}^\alpha$, $\mathcal{R}_\mathbf{s}^\alpha$ (the components of the friction force) and $\mathcal{R}_\mathrm{N}^\alpha$. The relative velocity $\boldsymbol{\mathcal{V}}^\alpha$ has for components $\mathcal{V}_\mathrm{T}^\alpha$, $\mathcal{V}_\mathbf{s}^\alpha$ (the components of the sliding velocity) and $\mathcal{V}_\mathrm{N}^\alpha$. *Actually the relative velocity involved in these laws is the right velocity* $\boldsymbol{\mathcal{V}}^{+\alpha}$, i.e. the velocity just after the considered instant of contact (where a possible velocity discontinuity occurs, for instance caused by a shock). In order to simplify the writing, we consider the two-dimensional case (we disregard the $\mathbf{s}$-components, and we omit the symbol $^\alpha$). The signed distance between the bodies is g (the gap).

The *inelastic shock law* reads as,

$$\text{if } g > 0 \text{ then } \mathcal{R}_\mathrm{N} = 0 \tag{9}$$

$$\text{if } g \leqslant 0 \text{ then } \mathcal{V}_\mathrm{N} \geqslant 0 \quad \mathcal{R}_\mathrm{N} \geqslant 0 \quad \mathcal{V}_\mathrm{N}\mathcal{R}_\mathrm{N} = 0 \tag{10}$$

The last relation, a complementary relation, implies that, when bodies are touching each other, the right component of the velocity is separating the bodies (since there is no adhesion), but if the normal component of the reaction is positive (some pressure is exerted), the normal component of the relative velocity vanishes, which means that the contacting bodies keep sticking or sliding on each other. Jean-Jacques Moreau has proven that these laws ensures impenetrability between bodies; see Jean-Jacques Moreau viability lemma [35,37,36].

The *Coulomb law* reads as,

$$|\boldsymbol{\mathcal{R}}_t| \leqslant \mu\mathcal{R}_n \quad \forall \boldsymbol{\mathcal{S}} \text{ such as } |\boldsymbol{\mathcal{S}}| \leqslant \mu\mathcal{R}_n \text{ then } \boldsymbol{\mathcal{V}}_t \cdot (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{R}}_t) \geqslant 0 \tag{11}$$

where $\mu \geqslant 0$ is the friction coefficient.

Both laws may be written in several equivalent manners in terms of Convex Analysis, for instance using the sub-differential of the indicatrix function of a proper convex set. The graphs of these laws are shown in Fig. 2. Those are graphs of multi mappings. Those laws being positively homogeneous, one is inclined to admit that they are still relevant when $\boldsymbol{\mathcal{R}}^\alpha$ is an impulse $\boldsymbol{\mathcal{I}}^\alpha$ instead of a force. Sometimes, this may be a strong modeling assumption.

In the case of deformable bodies, a restitution shock law may be irrelevant. Nevertheless, when a deformable body is numerically modeled, for instance using a finite element method, the configuration is described by node coordinates acting as materials points yielding a finite-dimensional mechanical system. The inelastic shock law is usually adopted. Coulomb's law is also relevant for flexible systems.

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Fig. 2. Inelastic shock and Coulomb graphs.

3.1.5. Discrete formulation of the dynamical equation

The time evolution is managed using an arbitrary time step $t_{n+1} = t_n + h$ of length h. Doing so, contact events are all captured and solved simultaneously during a time step. Integration of velocity (and external forces) during a time step is performed using a θ-scheme:

$$\mathbf{q}_{n+1} = \mathbf{q}_n + \int_{t_n}^{t_{n+1}} T(\mathbf{q})\mathbf{V}(\tau)\mathrm{d}\tau = \mathbf{q}_n + hT(\mathbf{q}_{n+\theta})\mathbf{V}_{n+\theta} \tag{12}$$

where $\mathbf{q}_{n+\theta} = \theta\mathbf{q}_{n+1} + (1-\theta)\mathbf{q}_n$ and $\mathbf{V}_{n+\theta} = \theta\mathbf{V}_{n+1} + (1-\theta)\mathbf{V}_n$. In practice, we also perform the integration of the rotation with an objective algorithm (see, for example, [38]).

Concerning granular materials, the time step and the spin are usually small, which makes the quadratic forces small or negligible. Therefore, this term is either omitted or explicitly integrated $\int_{]t_n,t_{n+1}]} \mathbf{F}_{\text{quad}}(\tau)\mathrm{d}\tau = h\mathbf{F}_{\text{quad}}(\mathbf{q}_m, \mathbf{V}_n)$ with $\mathbf{q}_m = \mathbf{q}_n + h(1-\gamma)\mathbf{V}_n, \gamma \in [0,1]$. However, in some situations with large time steps or high spin, it is more relevant to use an implicit generalized mid-point rule such as $\int_{t_n}^{t_{n+1}} \mathbf{F}_{\text{quad}}(\tau)\,\mathrm{d}\tau = h\mathbf{F}_{\text{quad}}(\mathbf{q}_{n+\theta}, \mathbf{V}_{n+\theta})$.

The mapping $\mathbb{H}^\alpha(\mathbf{q}_i, \mathbf{q}_j)$ is approximated as $\mathbb{H}^\alpha(\mathbf{q}_{m,i}, \mathbf{q}_{m,j})$. This is done using an explicit contact frame built with $\mathbf{q}_m$, so that local variables (gap, relative velocity, impulse) are expressed in this frame. When large displacements over a time step are observed, an implicit choice of the local frame is sometimes required. The formula (8) is discretized in a similar way:

$$g_{n+1}^\alpha = g_n^\alpha + \int_{t_n}^{t_{n+1}} \mathcal{V}_n^\alpha(t)\,\mathrm{d}t = g_n^\alpha + h(\theta\mathcal{V}_{\mathrm{N},n+1}^\alpha + (1-\theta)\mathcal{V}_{\mathrm{N},n}^\alpha) \tag{13}$$

Using the mappings between local and global variables ($\mathbb{H}$ and $\mathbb{H}^\top$) and the equation (3), the discrete form of the dynamical equation comes out as,

$$\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_{\text{free}} + \mathcal{W}\boldsymbol{\mathcal{I}} \quad \text{where: } \mathcal{W} = \mathbb{H}^\top(\mathbf{q}_m)\,\mathbb{M}^{-1}\,\mathbb{H}(\mathbf{q}_m) \tag{14}$$

where $\boldsymbol{\mathcal{V}}^\alpha$ represents now the *relative velocity at the end of the time step* playing the role of the right relative velocity and $\boldsymbol{\mathcal{I}}^\alpha$ is *the impulse over the time step*. The term $\boldsymbol{\mathcal{V}}_{\text{free}}$ accounts for all known data as given external forces. The matrix $\mathcal{W}$ is called the Delassus matrix. In other contexts and for rigid bodies, it is also called the mass matrix reduced to local variables.

Remark 2. In (6), $\mathbb{H}^\top$ may be viewed as an incidence matrix of the graph of all contacts. In other words, it possesses a non-null block $H^{\alpha,\top}(\mathbf{q}_i, \mathbf{q}_j)$ if a contact exists between bodies i and j. By construction, for a large collection of bodies, the matrix $\mathbb{H}^\top$ is sparse. Therefore, since $\mathbb{M}^{-1}$ is block diagonal (each block corresponding to a body), the matrix $\mathcal{W}$ is also block sparse and is not assembled in practice.

In the case of deformable bodies, one shows that using a θ-scheme (or any implicit time integration scheme) leads to the same numerical dynamical equation than for rigid bodies. Only the mass matrix is modified to obtain $\mathbb{M}_\theta = \mathbb{M} + \theta h\mathbb{C} + \theta^2 h^2 \mathbb{K}$.

3.1.6. Discrete formulation of the frictional contact law

The standard CD discrete frictional contact laws is a discrete formulation of (10), (11), and writes:

$$\text{if a contact is forecasted then } \mathcal{V}_{\mathrm{N}}^\alpha \geqslant 0 \quad \mathcal{I}_{\mathrm{N}}^\alpha \geqslant 0 \quad \mathcal{V}_{\mathrm{N}}^\alpha \mathcal{I}_{\mathrm{N}}^\alpha = 0$$

$$|\boldsymbol{\mathcal{I}}_{\mathrm{T}}| \leqslant \mu\mathcal{I}_{\mathrm{N}} \quad \forall \boldsymbol{\mathcal{S}} \text{ such as } |\boldsymbol{\mathcal{S}}| \leqslant \mu\mathcal{I}_{\mathrm{N}}, \text{ then } \boldsymbol{\mathcal{V}}_{\mathrm{T}} \cdot (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{I}}_{\mathrm{T}}) \geqslant 0$$

Both laws are summarized as

$$\text{law}(\boldsymbol{\mathcal{V}}^\alpha, \boldsymbol{\mathcal{I}}^\alpha) = 0 \tag{15}$$

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Remark 3. The proposition "if a contact is forecasted" means that at the considered candidates to contact α, bodies are sufficiently close (even more so bodies are numerically interpenetrating each other) according to some approximate explicit calculation of the gap, so that the inelastic shock law be processed. For instance, we can choose $g(q_m) \leqslant 0$ for activating the law. Otherwise, the reaction is null.

Some laws, such as restitution shock laws [35], adhesive laws with damage [39], viscoelastic contact laws involving the gap, may be considered as well. The kinematic (8) and its discrete form (13) allows us to write the frictional contact law under the form $\text{law}(\boldsymbol{\mathcal{V}}^\alpha, \boldsymbol{\mathcal{I}}^\alpha) = 0$ and to keep $\boldsymbol{\mathcal{V}}^\alpha, \boldsymbol{\mathcal{I}}^\alpha$ as the main unknowns.

3.1.7. Solving the standard CD system

The system to be solved is

$$\begin{aligned} &\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_{\text{free}} + \mathcal{W}\boldsymbol{\mathcal{I}} \\ &\forall \alpha \ \text{law}(\boldsymbol{\mathcal{V}}^\alpha, \boldsymbol{\mathcal{I}}^\alpha) = 0 \end{aligned}$$

The adopted formulation has allowed us to exhibit $\boldsymbol{\mathcal{V}}^\alpha$, *the relative velocity at the end of the time step*, and $\boldsymbol{\mathcal{I}}^\alpha$, *the impulse over the time step*, as main unknowns. The above system is solved using a NLGS method, performing loops (of iterate k) on the list of all α candidates to contact:

$$\begin{aligned} &\boldsymbol{\mathcal{V}}^{\alpha,k+1} = \boldsymbol{\mathcal{V}}^\alpha_{\text{free}} + \textstyle\sum_{\beta<\alpha} \mathcal{W}^{\alpha\beta}\boldsymbol{\mathcal{I}}^{\beta,k+1} + \sum_{\beta>\alpha} \mathcal{W}^{\alpha\beta}\boldsymbol{\mathcal{I}}^{\beta,k} + \mathcal{W}^{\alpha\alpha}\boldsymbol{\mathcal{I}}^{\alpha,k+1} \\ &\text{law}(\boldsymbol{\mathcal{V}}^{\alpha,k+1}, \boldsymbol{\mathcal{I}}^{\alpha,k+1}) = 0 \end{aligned} \tag{16}$$

A particular feature of the CD method is that for a single contact α, assuming provisional values of the local variables of neighboring candidates to contact ($\beta \neq \alpha$), the relative velocity and impulse are found straightforwardly, discussing the intersections of hyperplanes [40].

The loops are processed as follows:

(1) consider a contact α between two contacting bodies. Assume provisional values of the reactions on these bodies from neighboring contacts; those reactions are considered as external forces;
(2) the local-solver gives provisional values of the reaction at the contact α;
(3) all reactions are updated;
(4) the next contact in the list is processed;
(5) the list of contacts is run over and over until some convergence criteria is satisfied.

The convergence criteria uses distances to Signorini–Coulomb graphs. It happens that some laws, such as restitution shock laws, adhesive laws, viscoelastic contact laws, may be implemented using changes of variables, i.e. the new variables satisfy Signorini–Coulomb's laws. It is done by introducing two supplementary steps in the above loop, a change of variables and the reverse change.

3.2. Some applications

The CD method has been applied to investigate mechanical systems such as granular materials (rheology [41,42], tribology [43,44], industrial process [45,46]), masonry structures [47,48] or rock masses, under quasi-statics or dynamics loads, natural hazards (earthquake [49], landslide[50]), etc. More recently it was used to model robotics or multibody systems [51, 52]. Jean-Jacques Moreau, himself made 3D investigations into some controversial questions, such as segregation in shaken granular materials, pression under a heap of granular materials, shear bands in a 3-dimensional oedometric test, and also buildings made of blocks submitted to ground subsidence or earthquakes, some of these investigations were only privately published. One can say that a kind of maturity has been reached since the CD method is able to manage objects of any shape with various mechanical behavior and to take into account interaction laws as complex as necessary. Some examples are listed below:

- shape of objects may be described with simple convex primitives (disk, polygon, sphere, polyhedron, etc), compound of simple primitives or general triangulated surface convex or not. Contact detection has to be performed for any combination of primitives;
- bulk behavior of objects may be rigid as well as deformable (small or large transformation, using floating frame of reference) by means of the finite element method;
- a large set of interaction laws is available, e.g., frictional contact, cohesion (capillarity, damage, brittle fracture, etc.), wire, rod, etc.;
- multiple physics couplings (thermal effects, electrical conductivity, fluid dynamics, flow in porous media *etc*) are progressively taken into account. Physics couplings can be performed at different scales through upscaling/downscaling methods.

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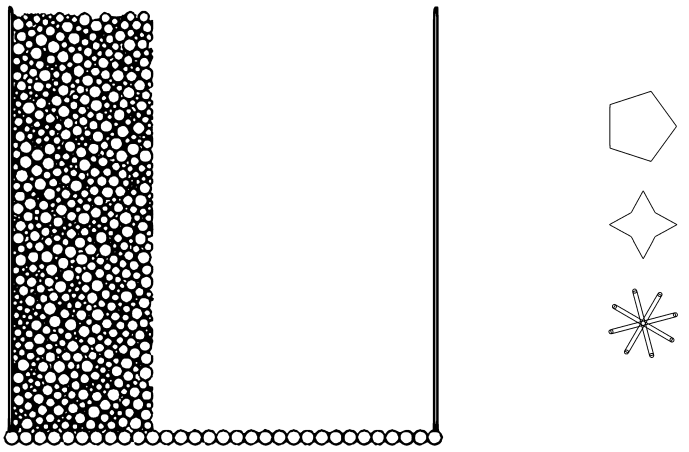

Fig. 3. Packing of a sample made of circles and shape of the grains.

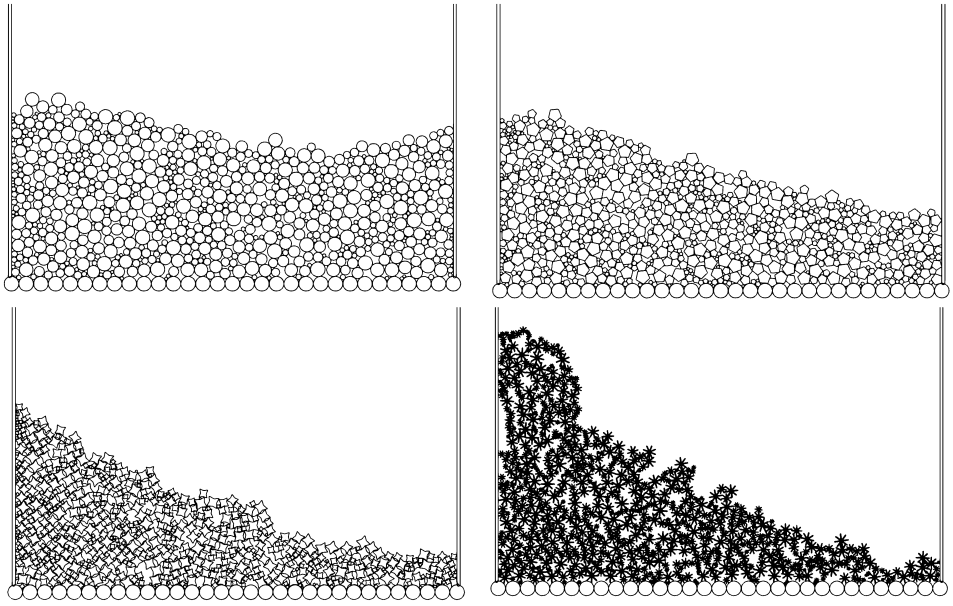

Fig. 4. Positions at rest of a collapsed column of grains. Influence of the objects' shape.

A first illustration concerns the collapse of a column of grains in a box (see Fig. 3) with a rough foundation. First a random granulometry of circles is generated. Then, using a geometrical circle packing algorithm, the position of the circles is computed. Finally, each circle is replaced by a disk, a pentagon (convex polygon), a star (non-convex polygon), or a flake (compound of rectangles and disks). Objects are considered as rigid. The coefficient of friction between objects is $\mu = 0.7$.

The positions at rest of samples made of disks, pentagons, stars or flakes are drawn in Fig. 4. Even with a quite high friction, the sample made of disks behaves like a fluid. Replacing disks by pentagons introduces some torque transmission and leads to arching and alignment phenomena. Replacing disks by non-convex stars introduces almost the same mechanisms than before, but it increases also the shear strength. Replacing disks by flakes introduces new bending phenomena due to tangle.

The second example concerns the cross section of a retaining wall, 1.5 m high, located in Largentière, France [53]. Fig. 5a displays the equilibrium configuration and the network of reaction forces arch-buttering the pressure of the ground exerted on the right-hand side of the wall (reactions are displayed as bold rectangles oriented in the direction of the forces, and the thicker the stronger). A 5-cm ground subsidence has been simulated as shown in Fig. 5b. The wall moves to a final equilibrium configuration (see Fig. 5c) with "a belly like" facing and typical open joints. The two bond stones contribute to this precarious equilibrium. When such bond stones are missing, the wall collapses in those circumstances.

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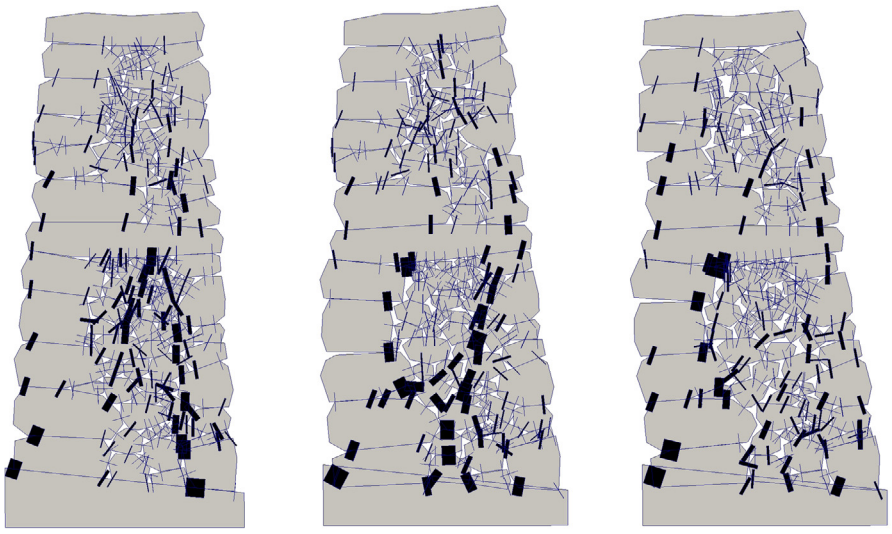

Fig. 5. Retaining wall: (a) equilibrium configuration, (b) ultimate ground subsidence, (c) equilibrium after ground subsidence.

4. Smoothed or nonsmooth methods: advantages and drawbacks

4.1. Interaction laws and numerical schemes

We have already quoted the family of DEM (Distinct Elements Methods). We come back shortly to the advantages and drawbacks of these methods, since some ideas may be commented from the point of view of both the DEM and the CD methods.

Choice of unknown variables. In DEM methods, the main unknowns are *the positions and the reaction forces* at the end of the time step, while in the CD method, the main unknowns are *the relative velocities at the end of the time step and the impulse reaction over the time step*. As we said before, this choice is motivated by the fact that impacts have to be consistently integrated.

Standard (explicit) schemes for ordinary differential equations requires smooth laws rather than nonsmooth laws require implicit integration. The DEM uses smoothed explicit schemes, while CD uses nonsmooth implicit schemes. Explicit schemes are easy to implement and some of them are very sophisticated and up to provide a high accuracy. To be able to write explicit schemes, the frictional contact laws must be expressed as standard (single-valued) mappings and must be differentiable enough according to the order of the scheme (reaction force versus gap for unilateral constraints, and friction force versus sliding velocity for the frictional laws). Those mappings are steep and yield stiff differential equations that are known to be difficult to integrate, especially for conserving stability and energy. For stability, it is necessary to use small time steps and some physical or numerical viscosity must be introduced. Furthermore, the fulfillment of the dissipation/conservation properties in discrete time is very difficult to ensure with explicit schemes and highly nonlinear contact models. Symplectic and geometric integration schemes [54] are not easy to design with switched nonlinear components. By the way, note that most of the compliant models are of limited differentiability, which prevents the use of higher-order schemes. In practice, people might use these schemes, but they have to face numerical instability and spurious oscillations. Thus conservative models are out of reach.

In the CD method the frictional contact laws are expressed as multi-mapping (shock laws and Coulomb law). The CD method is purely implicit for the nonsmooth terms. Though the choice of the local frames is usually done explicitly without loss of accuracy, the main unknowns, relative velocities and impulses, are computed within an implicit integration scheme. The reactions found at the end of a time step are those satisfying the dynamics, the shock law, and Coulomb's law during the time step. It can use large time steps, but each step needs a large number of NLGS iterations. They are many choices, but the larger the step, the more the number of iterations at each time step. In the CD method, the smooth nonlinearity (stiffness/damping, etc.) at contact is removed. This avoids stiff systems and we can ensure that the scheme respects some very important energetic conditions. Stability is then induced by these properties [55]. Note that these conservation and stability results are not available with a treatment of the contact condition at the position level.

Smooth elastic models or nonsmooth rigid models. It is the place to get rid of a controversial discussion: it is said that shock laws and Coulomb's law are not physical because they disregard local deformability; on the other side, it is said that, since

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local deformability laws are hardly known or measurable, shock laws and Coulomb law account satisfactorily for the main features of frictional contact laws (unilateral behavior and sliding threshold). This controversial discussion is irrelevant. The choice is a matter of modeling and of numerical facilities. In any numerical method, the choice of the numerical parameters must be appropriate so as to ensure a correct modeling. For instance, in the DEM, the local stiffness of the contacting bodies is considered so as to ensure a moderate interpenetration. This stiffness is thus quite large, which consequently leads to use quite small time steps in order to secure the stability of explicit numerical schemes. When the local physics is restricted only to elasticity, waves propagate, which in many applications, in soil mechanics for instance, is of no interest. Looking for steady states with DEM methods, the local physics should be augmented with viscosity contents, of questionable values. Anyway, any damping from physical considerations or any numerical trick may do, if properly scaled. Quasi-static and equilibrium conditions are really difficult to reach with compliant models. The compliant methods often calls for "cosmological damping". Although these methods claim a more accurate mechanical modeling, such numerical tricks are not acceptable. Finally, with a nonlinear compliant model with damping, it is very difficult to get nearly plastic impacts. In other works, low values of the coefficient of restitution are not reachable. Except for bi-stiffness models and ideal elasto-plastic models, that are, by the way nonsmooth, it is impossible to get large dissipation rates with smooth compliant contact models (see [56], for instance).

Velocity level constraints implies interpenetration, but corrections are possible. A possible drawback of the CD method, with respect to the idea of a fully implicit formulation, is the fact that the gap between the contacting bodies needs to be evaluated explicitly to activate the constraints at the velocity level (see Remark 3). This could be done implicitly, but numerical experiments show that it is not worthwhile. A similar drawback comes from the way the unilateral constraints are modeled. It is said that when a contact is occurring, the normal component of the relative velocity must be positive after the impact. This formulation introduced by Jean-Jacques Moreau [57] ensures that the gap remains positive in a continuous time evolution. Nevertheless, in a time stepping integration scheme, this formulation provides approximate gaps and may generate interpenetrations, sometimes self-restoring. There exist at least three methods to circumvent this issue.

- Some *"viscoelastic device"* may be introduced between contacting bodies, device governed by an inelastic shock law and Coulomb law, after a convenient change of variables [32–34]. This amounts to introducing some additional terms in the Delassus matrix improving the conditioning. The same CD method is thus used to perform numerical computations, with large time steps and a number of iterations at each time step less than usual. The oscillator composed of two contacting bodies linked by the viscoelastic device is "critical" in the sense that the stiffness and viscosity values are such that the free oscillations are damped. In fact, the relevant parameter is the period of oscillations, which is currently chosen equal to the time step. The proposed device should be considered as a regularizing artifact rather than a real model of the local viscoelasticity as in the DEM. It prevents waves propagation and interpenetrations are self restoring. In differential algebraic equations, a usual tool to stabilize the constraints is to introduce a critical harmonic oscillator (the "device") in the joints. This technique is known as the Baumgarte stabilization. This device may be viewed as a nontrivial extension of the Baumgarte stabilization technique to unilateral constraints.
- Another possible solution in the case of inelastic shock law is to impose the satisfaction of the position and velocity constraints over two steps, as is has been developed in [32].
- Finally, a solution to cope with interpenetration is to use a velocity formulation of the non-interpenetration law, but introducing an additional constraint at the position level to recover a null value of the gap as proposed by Acary [28] or Brüls et al. [52]. This latter method is an adaptation of the well-know Gear–Gupta–Leimkühler technique for the stabilization of differential algebraic equations.

Indeterminacy and slow convergence of the NLGS method. Another drawback of the CD method is the slow convergence of the NLGS algorithm. Actually, the NLGS solver is not to blame and it may be improved as mentioned hereafter. The slow convergence comes from indeterminacy (plurality of solutions). By indeterminacy we mean, for instance in the case of a sample at equilibrium, that there is not a unique set of reactions achieving the equilibrium, since there are kinematic constraints. To figure out roughly the situation, we may propose the following analogy: when bodies are linked together by springs, the system has a single equilibrium state (let us say that the system is isostatic). It is the case of the DEM. It is not the case of the CD method where there are rigid connections between bodies (the system is hyperstatic); it is a case of global indeterminacy. When two polygonal objects meet, a side of a polygon set onto a side of the other, the situation is handled choosing two contact loci where "nodal" reactions are exerted. This localization process generates indeterminacy; it is a case of local indeterminacy. From a mathematical point of view, when indeterminacy occurs, the mapping $\mathbb{H}$ has a kernel that is not reduced to $\{0\}$. It is the set of self-equilibrated reactions. For example, for rigid contacts $\dim(\ker \mathbb{H}) \sim d.o.f._{\text{contacts}} - d.o.f._{\text{objects}}$. It happens that the algorithm has difficulty to select an element in the kernel and it has no reason to do it, except that the initializing conditions may incline the algorithm to select some value. Concerning local indeterminacy, one can introduce some ad hoc relationship between contacts to reduce the kinematic constraints [58]. Introducing the viscoelastic device mentioned above theoretically annihilates the indeterminacy, and thus improves the convergence. Other attempt may be found in [59,60].

When interested in macroscopic data as the resultant of reactions forces on a wall, the algorithm gives reliable values. In fact, a distribution of contact forces is selected by the algorithm [61] and it appears that the network of weak reactions

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forces is sensitive to the way the algorithm proceeds, while the network of strong reactions forces is rather steady. The effect of indeterminacy is more acute in monodisperse collections, we mean samples of grains with almost the same radius. It is also the case in monuments made of blocks, for instance a wall built with the same blocks. The solution yielded by the algorithm, though mechanically true, may look strange, for instance being quite asymmetric. Scrambling, i.e. changing at random the order of the contacts in the NLGS iterations improves the look of the selected solution. Using a Jacobi solver also helps. An efficient solution is to use the viscoelastic device described above. It allows us to select a sensible solution and it reduces the number of iterations.

However, the NLGS procedure has also drawbacks. Some of them are related to the internal operation of the method, and particularly the way information propagates. It may be improved using a relaxation procedure. In some situations, the NLGS procedure can be replaced by a faster solver [62,63]. Otherwise, parallel computing can help to obtain some reasonable computation time [64,65].

4.2. Bulk behavior: floating frame of reference

For several reasons detailed hereafter, using deformable-bodies instead of rigid ones may offer numerical and modeling advantages. As previously explained, indeterminacy may slow down the convergence rate of NLGS procedure. Increasing the number of degrees of freedom of bodies makes $\mathbb{H}$ a full-rank matrix. Concerning the shock law, using a Newton restitution law is questionable [66,67], it is able to mimic effects of wave propagation and dissipation, but only for binary contacts. Using deformable bodies should overcome partly this issue. In any case, managing multiple impacts remains an open problem [68,69]. Regarding mechanical analysis, fracture mechanics, upscaling techniques, etc., the computation of a stress or a deformation field in a discrete material is of main interest. The so-called Weber tensor is an approximation of the Cauchy stress tensor on a gauge [70] for a quasi-static deformation,

$$<\underline{\underline{\sigma}}>=\frac{1}{V}\sum_{\alpha=1}^{n_c}\mathbf{l}^{\alpha}\otimes\boldsymbol{\mathcal{R}}^{\alpha} \tag{17}$$

where V is the volume of the gauge, n_c the number of contacts in the gauge, $\boldsymbol{\mathcal{R}}$ are the reactions between contacting bodies, and $\mathbf{l}$ are the branch vectors between the center of the gauge and the contact points. There exists several attempts revisiting this formula [71–73]. However, using such formula on a unique body is questionable due to indeterminacy. In such situation, using deformable-body models gives naturally access to a stress field. Finally, it may be necessary to introduce some structural effects in bodies to catch phenomena such as compression or flexion, waves, etc., where once again rigid-body models are too rough.

Modeling collections of deformable-bodies or mixture of deformable bodies and rigid bodies was already done with CD/NLGS [32,47,74,75,46]. However, due to kinematic non-linearities, such approaches are really CPU time consuming, except in very particular cases, since bodies are subject to large rotations whatever the deformation.

The main idea to overcome this issue is to consider the body motion as the sum of a rigid-body motion and a deformable-body motion relatively to the rigid-body motion. This deformable-body motion is supposed to be sufficiently small such that the center of inertia of the body remain the same (small perturbations).

The present work is inspired by ideas developed in Fraeijs de Veubeke [76], Shabana [77], Acary [47], Felippa [78] or Koziara [79].

Motion can be described through:

- The rigid-body motion of the center of inertia (M_c) which is defined by: generalized coordinates $\mathbf{q}^R=\{\mathbf{x}_c,\mathcal{R}_c\}^\top$, where $\mathbf{x}_c$ is the position and $\mathcal{R}_c$ is the orientation matrix of the body and generalized velocity $\mathbf{v}^R=\{\mathbf{v}_c,\boldsymbol{\Omega}_c\}^\top$, where $\mathbf{v}_c$ is the translation velocity and $\boldsymbol{\Omega}_c$ the angular velocity. Note that $\dot{\mathcal{R}}_c=\mathcal{R}_c\tilde{\boldsymbol{\Omega}}_c$.
- A deformable-body displacement/velocity relatively to the rigid-body motion.

The velocity field $\mathbf{v}$ splits into a rigid-body velocity $\mathbf{v}_R$ and a deformable-body one $\mathbf{v}_D$ as:

$$\mathbf{v}(\mathbf{x})=\underbrace{\mathbf{v}_c+\boldsymbol{\Omega}_c\times(\mathbf{x}-\mathbf{x}_c)}_{\mathbf{v}_R(\mathbf{x})}+\mathbf{v}_D(\mathbf{x}) \tag{18}$$

Velocity field decomposition is based on two mappings:

- A distribution mapping G which interpolates the rigid-body motion at each point knowing the generalized rigid-body velocity: $\mathbf{v}_R(\mathbf{x})=G(\mathbf{x},\mathbf{v}^R)$.
- A projection mapping L which extracts the generalized rigid-body velocity from any velocity field. $L(G(\bullet))$ is identity on the image of G mapping, noted $\mathcal{V}_R$.

Operators are chosen to split the velocity vector space $\mathcal{V}$ into a direct sum of vector spaces related to pure deformable-body velocities $\mathcal{V}_D$ and rigid-body velocities $\mathcal{V}_R$. By duality, the forces space $\mathcal{F}$ splits into two supplementary spaces $\mathcal{F}_R$ and $\mathcal{F}_D$.

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$$\begin{array}{ccc} \mathcal{V} = \mathcal{V}_{\mathrm{R}} \oplus \mathcal{V}_{\mathrm{D}} & <,> & \mathcal{F} = \mathcal{F}_{\mathrm{R}} \oplus \mathcal{F}_{\mathrm{D}} \\ L \downarrow\uparrow G & & G^{\top} \downarrow\uparrow L^{\top} \\ \mathcal{V}^{R} & <.> & \mathcal{F}^{R} \end{array} \tag{19}$$

A possible way to build G and L operators is:

$$G(\mathbf{x}, \mathbf{v}^{R}) = \mathbf{v}_{\mathrm{R}}(\mathbf{x}) = \mathbf{v}_{\mathrm{c}} + \boldsymbol{\Omega}_{\mathrm{c}} \times (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \tag{20}$$

$$L(\mathbf{v}) = \begin{pmatrix} \mathbf{v}_{\mathrm{c}} = m^{-1} \int_{\Omega} \rho\, \mathbf{v}\, \mathrm{d}\Omega \\ \boldsymbol{\Omega}_{\mathrm{c}} = \mathbb{J}^{-1} \int_{\Omega} \rho (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \times \mathbf{v}\, \mathrm{d}\Omega \end{pmatrix} \tag{21}$$

where:

- ρ is the density of the solid,
- $m = \int_{\Omega} \rho(\mathbf{x})\, \mathrm{d}\Omega$ is the total mass; $\mathbb{M} = m\,\mathbb{I}$.
- $\mathbb{J} = \int_{\Omega} (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \cdot (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \mathbb{I} - (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \otimes (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \rho(\mathbf{x}) \mathrm{d}\Omega$ is the inertia.

A smooth motion of the body is governed by the momentum conservation law:

$$\rho(\mathbf{x}, t) \dot{\mathbf{v}}(\mathbf{x}, t) = \mathrm{div}(\underline{\underline{\sigma}}(\mathbf{x}, t)) + \rho(\mathbf{x}, t) \mathbf{f}_{v}(t), \forall \mathbf{x} \in \Omega(t) \tag{22}$$

where $\mathbf{f}_{v}$ is the body forces, $\dot{\mathbf{v}}$ the acceleration and $\underline{\underline{\sigma}}$ the stress tensor. Surface forces T are imposed on Γ_{f}, which normal is N ($\mathrm{T} = \underline{\underline{\sigma}} \cdot \mathrm{N}$). Velocities are imposed on Γ_{v}.

Considering the velocity form introduced in (18), the acceleration $\dot{\mathbf{v}}$ takes the following expression:

$$\dot{\mathbf{v}} = \dot{\mathbf{v}}_{\mathrm{c}} + \dot{\boldsymbol{\Omega}}_{\mathrm{c}} \times (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) + \boldsymbol{\Omega}_{\mathrm{c}} \times (\boldsymbol{\Omega}_{\mathrm{c}} \times (\mathbf{x} - \mathbf{x}_{\mathrm{c}})) + \boldsymbol{\Omega}_{\mathrm{c}} \times \mathbf{v}_{\mathrm{D}} + \dot{\mathbf{v}}_{\mathrm{D}} \tag{23}$$

In the following, a virtual velocity field is considered such that $\delta\mathbf{v}(\mathbf{x}) \in \delta\mathcal{V}^{0}$, with:

$$\delta\mathcal{V}^{0} = \{\delta\mathbf{v}(\mathbf{x}) = \delta\mathbf{v}_{\mathrm{c}} + \delta\boldsymbol{\Omega}_{\mathrm{c}} \times (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) + \delta\mathbf{v}_{\mathrm{D}}(\mathbf{x}), \forall \mathbf{x} \in \Omega(t) \text{ and homogeneous for fixed d.o.f.} \tag{24}$$

Starting from (22), the principle of virtual power is derived and one recovers a set of three coupled equations:

- Newton's equation:

$$\begin{aligned} & m\dot{\mathbf{v}}_{\mathrm{c}} + \int_{\Omega(t)} \rho(\mathbf{x}) \dot{\mathbf{v}}_{\mathrm{D}}\, \mathrm{d}\Omega \\ & \quad = \int_{\Omega(t)} \rho(\mathbf{x}) \mathbf{f}_{v}(t)\, \mathrm{d}\Omega + \int_{\Gamma(t)} \mathrm{T}(\mathbf{x}, t)\, \mathrm{d}\Gamma \end{aligned} \tag{25}$$

- Euler's equation:

$$\begin{aligned} & \mathbb{J}\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbb{J}\boldsymbol{\Omega} + \int_{\Omega(t)} \rho(\mathbf{x}) (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \times \dot{\mathbf{v}}_{\mathrm{D}}(\mathbf{x}, t) \mathrm{d}\Omega \\ & \quad = \int_{\Omega(t)} \rho(\mathbf{x}) (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \times \mathbf{f}_{v}(t)\, \mathrm{d}\Omega + \int_{\Gamma(t)} (\mathbf{x} - \mathbf{x}_{\mathrm{c}}) \times \mathrm{T}(\mathbf{x}, t)\, \mathrm{d}\Gamma \end{aligned} \tag{26}$$

- Deformable-part equation:

$$\begin{aligned} & \int_{\Omega(t)} \rho(\mathbf{x}) \dot{\mathbf{v}}_{\mathrm{D}} \cdot \delta\mathbf{v}_{\mathrm{D}}\, \mathrm{d}\Omega + \boldsymbol{\Omega} \times \int_{\Omega(t)} \rho(\mathbf{x}) \mathbf{v}_{\mathrm{D}}(\mathbf{x}, t) \cdot \delta\mathbf{v}_{\mathrm{D}}\, \mathrm{d}\Omega \\ & \qquad = -\int_{\Omega(t)} \underline{\underline{\sigma}}(\mathbf{x}, t) : \nabla\delta\mathbf{v}_{\mathrm{D}}(\mathbf{x}, t)\, \mathrm{d}\Omega(\mathbf{x}, t) + \int_{\Omega(t)} \rho(\mathbf{x}) \mathbf{f}_{v}(t) \cdot \delta\mathbf{v}_{\mathrm{D}}\, \mathrm{d}\Omega \\ & \qquad + \int_{\Gamma_{f}(t)} \mathrm{T}(\mathbf{x}, t) \cdot \delta\mathbf{v}_{\mathrm{D}}\, \mathrm{d}\Gamma \end{aligned} \tag{27}$$

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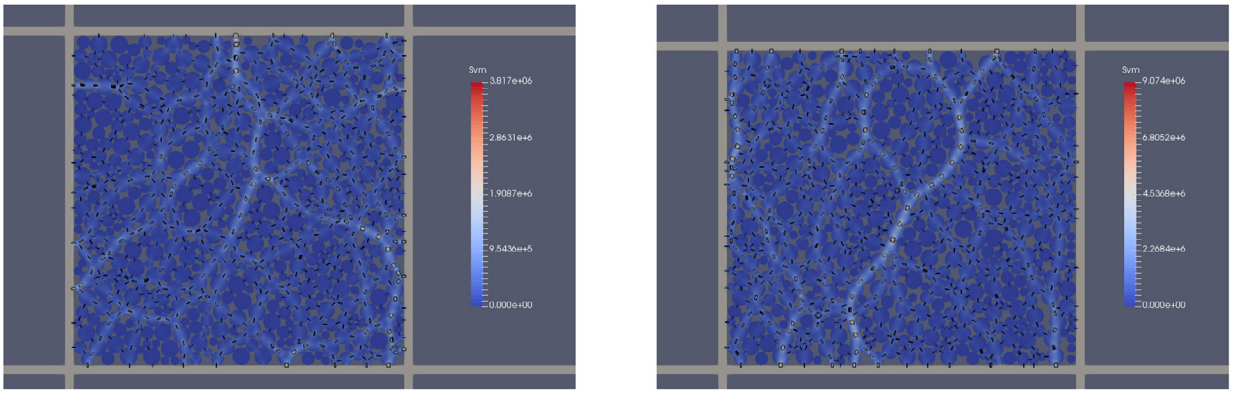

(a) (b)

Fig. 6. Bi-axial test, Von Mises stress evolution: (a) hydrostatic compression configuration, (b) vertical compression configuration $\varepsilon = 4\%$.

Introducing quantities expressed in the floating frame: $\mathbf{x}'$ the position, $\boldsymbol{\Omega}'$ the spin vector ($\boldsymbol{\Omega}_c = \mathcal{R}_c\boldsymbol{\Omega}'$) and $\mathbf{v}'_D$ the deformable velocity, one obtains:

$$\begin{cases} m\dot{\mathbf{v}}_c + \mathcal{R}_c \int_{\Omega(t)} \rho(\mathbf{x}')\dot{\mathbf{v}}'_D \,\mathrm{d}\Omega = \int_{\Omega(t)} \rho(x)\mathbf{f}_v(t)\,\mathrm{d}\Omega + \int_{\Gamma(t)} \mathrm{T}(x,t)\,\mathrm{d}\Gamma \\ \mathbb{J}'\dot{\boldsymbol{\Omega}}' + \boldsymbol{\Omega}' \times \mathbb{J}'\boldsymbol{\Omega}' + \int_{\Omega(t)} \rho(\mathbf{x}')\mathbf{x}' \times \dot{\mathbf{v}}'_D \,\mathrm{d}\Omega = \\ \qquad \int_{\Omega(t)} \rho(\mathbf{x}')\mathbf{x}' \times \mathcal{R}_c^\top \mathbf{f}_v(t)\mathrm{d}\Omega + \int_{\Gamma(t)} \mathbf{x}' \times \mathcal{R}_c^\top \mathrm{T}(\mathbf{x},t)\,\mathrm{d}\Gamma \\ \int_{\Omega(t)} \rho(\mathbf{x}')\dot{\mathbf{v}}'_D \cdot \delta\mathbf{v}'_D \mathrm{d}\Omega + 2\boldsymbol{\Omega}' \times \int_{\Omega(t)} \rho(x')\mathbf{v}'_D \cdot \delta\mathbf{v}'_D \mathrm{d}\Omega + \int_{\Omega(t)} \underline{\underline{\sigma}}' : \nabla\delta\mathbf{v}'_D \,\mathrm{d}\Omega = \\ \qquad \int_{\Omega(t)} \rho(\mathbf{x})\mathcal{R}_c^\top \mathbf{f}_v(t) \cdot \delta\mathbf{v}'_D \,\mathrm{d}\Omega + \int_{\Gamma(t)} \mathcal{R}_c^\top \mathrm{T}(\mathbf{x},t) \cdot \delta\mathbf{v}'_D \,\mathrm{d}\Gamma \end{cases} \tag{28}$$

In the floating reference frame, the stiffness matrix is constant. $\mathbb{W} = (\mathbb{M}' + h^2\theta^2\mathbb{K}')^{-1}$ and is computed once for all. One only have to rotate forces and velocities from the global to the floating frame. Omitting the deformable-body contribution, one recovers the classical Newton–Euler formulation. We refer to [80] for additional discussions on this model.

4.3. Example

Here are presented the results of a bi-axial test (2D) made of 500 deformable disks computed with the floating frame of reference method. For the sake of clarity, we have only presented a 2D example, even if the method works also in 3D.

A set of radii is generated picking randomly in $\{0.1\,\mathrm{m}, 0.15\,\mathrm{m}, 0.2\,\mathrm{m}\}$. Each grain is meshed using triangles with the same reference size (0.025 m). It implies that the boundaries of the grains are no more smooth curves but poly-lines. The mechanical parameters are the following: density $\rho = 2500\,\mathrm{kg/m^3}$, Young modulus $E = 50$ GPa and Poisson coefficient $\nu = 0.3$. First, a hydrostatic pressure of 0.1 MPa is applied through a driven force prescribed on the top and right walls (Fig. 6a). Then a vertical deformation is applied imposing a velocity on the top wall while the horizontal pressure is maintained to 0.1 MPa (Fig. 6b). The left and down walls are clamped.

One obtains the stress field in the grains; in Fig. 6, the Von Mises stress norm is drawn after the hydrostatic compression phase (6a) and at the end of the following vertical compression phase (6b). The intensity of the stress is clearly related to the contact forces. The vertical stress–strain curve is depicted in Fig. 7a. It is compared to the curve obtained with rigid disks, with rigid meshed objects (omitting the $\mathbf{v}_D$ contribution) and taking a high value of the Young modulus ($E = 1\mathrm{e}14$ Pa). Obviously, one observes that the rigid models are not able to catch the elastic part of the behavior. In Fig. 7b, one observes that, for a given tolerance of the NLGS method, using deformable bodies the contact solver needs less iterations to converge.

5. Conclusion

Several years ago, Jean-Jacques Moreau built a bridge between theoretical tools of convex analysis and applications thanks to the CD method. His original works, introducing a rigorous and robust framework, open the way for scientists to tackle a huge set of problems. Such works are still in progress. The CD method has demonstrated to be a relevant framework for the numerical treatment of frictional contact problems (including several behaviors such as cohesive or viscoelastic laws), but also ad hoc laws (bilateral constraint, wire, spring, etc.). Due to its implicit formulation, contact laws are simultaneously taken into account with dynamics equations. Obviously, set-valued conditions can be managed in an exact form, i.e. without introducing any regularization. Furthermore, the CD method enjoys very good stability properties due to its implicit time

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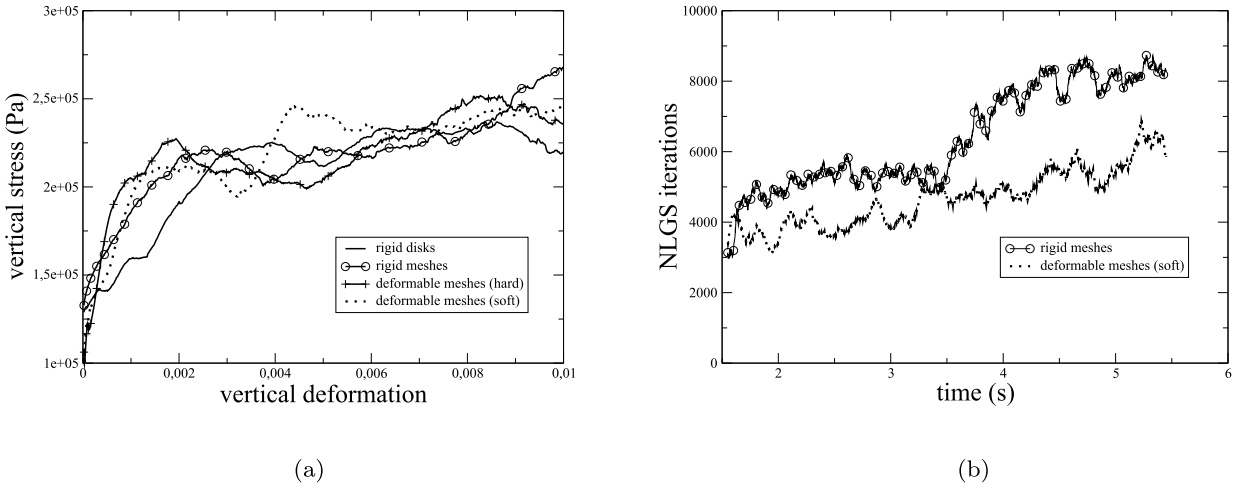

Fig. 7. Bi-axial test: (a) stress–strain curve, (b) NLGS iterations evolution during the compression phase.

integration scheme. However, the uniqueness of the reactions is not guaranteed though the uniqueness of velocities and displacements is often observed. Since the contact law does not depend on the body shape, managing any shape is a generic feature of the CD method, which is an advantage considering force laws used in DEM. Several variants of the method were developed to take into account deformability or to manage parallel computing. Since deformability is taken into account by body internal degrees of freedom in the CD method, rather than at the contact level in DEM, the roles played by bulk and contact behaviors can be analyzed. Latest works are related to multiple physics couplings.

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